

EMBEDDING PROPERTIES OF HEREDITARILY JUST INFINITE PROFINITE WREATH PRODUCTS

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ABSTRACT. We study infinitely iterated wreath products of finite permutation groups w.r.t. product actions. In particular, we prove that, for every non-empty class of finite simple groups \mathcal{X} , there exists a finitely generated hereditarily just infinite profinite group W with composition factors in \mathcal{X} such that any countably based profinite group with composition factors in \mathcal{X} can be embedded into W . Additionally we investigate when infinitely iterated wreath products of finite simple groups w.r.t. product actions are co-Hopfian or non-co-Hopfian.

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. A profinite group G is just infinite if G is infinite and every non-trivial closed normal subgroup $N \trianglelefteq_c G$ is open in G . While a complete classification of just infinite profinite groups is way out of reach, there is a natural interest in understanding as much about their structure as possible. It is known (e.g., see [7, Theorem 3]) that every just infinite profinite group either is a profinite branch group or contains an open subgroup isomorphic to the direct product of a finite number of copies of a hereditarily just infinite profinite group, where a profinite group G is called *hereditarily just infinite* if every open subgroup $H \leq_o G$ is just infinite. While branch groups have been studied quite extensively (e.g., see [2]) comparatively little is known about hereditarily just infinite groups.

Well-known families of hereditarily just infinite profinite groups are supplied by compact open subgroups of simple algebraic groups over non-archimedean local fields, e.g., groups such as $\mathrm{SL}_n(\mathbb{Z}_p)$ or $\mathrm{SL}_n(\mathbb{F}_p[[t]])$; see [8]. In addition there are some ‘sporadic’ non-linear examples, such as $\mathrm{Aut}(\mathbb{F}_p[[t]])$ and certain subgroups thereof; see [3, 1, 5]. In [16, Theorem A], J. S. Wilson gave the first examples of hereditarily just infinite profinite groups that are not virtually pro- p for any prime p . They arise as certain iterated wreath products of non-abelian finite simple groups, and retrospectively the construction is very flexible. In [16, 10, 14] some embedding, generation and presentation properties

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of such groups have been established, but many of their features are not yet fully understood. In passing, we remark that A. Lucchini has used crown-based powers to manufacture further examples of hereditarily just infinite profinite groups; see [9]. Interesting new types of hereditarily just infinite pro- p groups were constructed by Ershov and Jaikin in [6].

In this paper we focus on hereditarily just infinite profinite groups that are obtained as inverse limits of iterated wreath products w.r.t. product actions. They arise as follows; see Section 2 for a more detailed description. Let $\mathcal{S} = (S_k)_{k \in \mathbb{N} \cup \{0\}}$, with $S_k \leq \text{Sym}(\Omega_k)$, be a sequence of finite transitive permutation groups. The inverse limit

$$W^{\text{pa}}(\mathcal{S}) = \varprojlim W_n^{\text{pa}}$$

of the inverse system $W_0^{\text{pa}} \leftarrow W_1^{\text{pa}} \leftarrow \dots$ of finite iterated wreath products w.r.t. product actions

$$W_n^{\text{pa}} = S_n \circledast (S_{n-1} \circledast (\dots \circledast S_0)) \leq \text{Sym}(\hat{\Omega}_n) \quad \text{for } \hat{\Omega}_n = \Omega_n^{(\Omega_{n-1}^{(\dots^{\Omega_0})})}.$$

is called the *infinitely iterated wreath product of type \mathcal{S} w.r.t. product actions*.

By [11, Theorem 6.2] and [10], every infinitely iterated wreath product w.r.t. product actions $W^{\text{pa}}(\mathcal{S})$, based on a sequence \mathcal{S} of finite non-abelian simple permutation groups, is a finitely generated hereditarily just infinite profinite group that is not virtually pro- p for any prime p .

1.2. Main results. The aim of this paper is to study embedding properties of infinitely iterated wreath product of finite non-abelian simple groups w.r.t. product actions. Specifically, we are interested in embeddings of countably based profinite groups with specified (topological) composition factors into such wreath products. By [16, Theorem A] and [10], there exists a finitely generated hereditarily just infinite profinite group \mathcal{G} such that every countably based profinite group can be embedded into \mathcal{G} as a closed subgroup. Our first theorem is a refinement of this result to profinite groups with restricted composition factors. Recall that, by virtue of the Jordan–Hölder Theorem for finite groups, every countably based profinite group G has a countable set of composition factors with well-defined multiplicities; cf. Section 2.

Theorem A. *Let $\mathcal{S} = (S_n)_{n \in \mathbb{N} \cup \{0\}}$ be a sequence of finite simple groups. Then every profinite group G that admits a composition series*

$$G = G_1 \triangleright G_2 \triangleright \dots \quad \text{with factors } G_k/G_{k+1} \cong S_k, k \in \mathbb{N},$$

embeds as a closed subgroup into the infinitely iterated wreath product $W^{\text{pa}}(\mathcal{S})$ of type $\mathcal{S} = (S_k)_{k \in \mathbb{N} \cup \{0\}}$ w.r.t. product actions, where each $S_k \leq \text{Sym}(S_k)$ acts regularly on itself by right multiplication.

We emphasise that Theorem A includes the possibility of some of the simple groups S_k being cyclic. Making further adjustments, we construct for any given class of finite simple groups \mathcal{X} an infinitely iterated wreath product $W^{\text{pa}}(\mathcal{S}_{\mathcal{X}})$ with composition factors in \mathcal{X} that satisfies a ‘universal property’ for embedding countably based profinite group with composition factors in \mathcal{X} . The construction is flexible and the resulting group is in general not unique.

Corollary B. *Let \mathcal{X} be a non-empty class of finite simple groups. Then there exists a sequence $\mathcal{S}_{\mathcal{X}} = (S_k)_{k \in \mathbb{N}}$ of groups $S_k \in \mathcal{X}$, where each $S_k \leq \text{Sym}(S_k)$ acts regularly on itself by right multiplication, such that every countably based profinite group with composition factors in \mathcal{X} embeds as a closed subgroup into the infinitely iterated wreath product $W^{\text{pa}}(\mathcal{S}_{\mathcal{X}})$ of type $\mathcal{S}_{\mathcal{X}}$.*

Our proof of Theorem A relies on an apparently little known construction to embed iterated wreath products w.r.t. imprimitive actions into iterated wreath products w.r.t. product actions; see Proposition 3.3 and its Corollary 3.4.

Furthermore we are interested in when infinitely iterated wreath products of finite simple groups w.r.t. product actions are or fail to be *co-Hopfian*. Recall that a profinite group G is *co-Hopfian*, if there exists no proper closed subgroup $H \subsetneq G$ with $H \cong G$. Our description of non-co-Hopfian groups relies on the concept of ‘permutational isomorphism’ of permutation groups; compare [4, p. 17] and see Definition 3.2 for a natural generalisation.

Definition. We say that a permutation group $H \leq \text{Sym}(\Delta)$ is *permutationally isomorphic to a subgroup* of a permutation group $G \leq \text{Sym}(\Omega)$ if there exist $\tilde{H} \leq G$ and an \tilde{H} -invariant subset $\tilde{\Delta} \subseteq \Omega$ such that $H \leq \text{Sym}(\Delta)$ is equivalent to the faithfully induced permutation group $\tilde{H}|_{\tilde{\Delta}} \leq \text{Sym}(\tilde{\Delta})$: there exist a group isomorphism $\iota: H \rightarrow \tilde{H}$ and a bijection $\gamma: \Delta \rightarrow \tilde{\Delta}$ such that $\gamma(\delta^h) = \gamma(\delta)^{\iota(h)}$ for all $\delta \in \Delta$ and $h \in H$. For instance, the permutation group $H = \langle (1\ 2) \rangle \leq \text{Sym}(2)$ is permutationally isomorphic to a subgroup of $G = \langle (1\ 2)(3\ 4) \rangle \leq \text{Sym}(4)$ via $\tilde{H} = G$ and $\tilde{\Delta} = \{3, 4\}$.

Observe that the relation “permutationally isomorphic to a subgroup” on permutation groups is transitive. We say that the terms of a sequence $\mathcal{S} = (S_k)_{k \in \mathbb{N}}$ of finite permutation groups $S_k \leq \text{Sym}(\Omega_k)$ are *eventually permutationally isomorphic to subgroups of later terms*, if there exists $n_0 \in \mathbb{N}$ such that, for every $j \in \mathbb{N}_{\geq n_0}$, there is at least one (equivalently: there are infinitely many) $k \in \mathbb{N}_{> j}$ for which $S_j \leq \text{Sym}(\Omega_j)$ is permutationally isomorphic to a subgroup of $S_k \leq \text{Sym}(\Omega_k)$.

We establish the following results.

Theorem C. *Let $\mathcal{S} = (S_k)_{k \in \mathbb{N}}$ be a sequence of non-trivial finite permutation groups $S_k \leq \text{Sym}(\Omega_k)$. If the terms of \mathcal{S} are eventually permutationally isomorphic to subgroups of later terms, then the infinitely iterated wreath product $W^{\text{pa}}(\mathcal{S})$ of type \mathcal{S} is non-co-Hopfian.*

Theorem C applies, in particular, to constant sequences of finite simple groups, but also to the sequence $\mathcal{S} = (S_k)_{k \in \mathbb{N}}$ of pairwise non-isomorphic alternating groups $S_k = \text{Alt}(k+4) \leq \text{Sym}(k+4)$.

For our final result, recall that a finite group S is *minimal non-abelian simple* if it is non-abelian simple and every proper subgroup of S is soluble; such groups were classified by J. G. Thompson [12, 13] well before the classification of all finite simple groups.

Corollary D. *Let $\mathcal{S} = (S_k)_{k \in \mathbb{N}}$ be a sequence of finite transitive permutation groups $S_k \leq \text{Sym}(\Omega_k)$ that are minimal non-abelian simple. Then the infinitely iterated wreath product $W^{\text{pa}}(\mathcal{S})$ is non-co-Hopfian if and only if the terms of \mathcal{S} are eventually permutationally isomorphic to subgroups of later terms.*

Observe that, if \mathcal{S} consists of minimal non-abelian simple groups, then the terms of \mathcal{S} are eventually permutationally isomorphic to subgroups of later terms if and only if almost all terms occur infinitely often in \mathcal{S} .

To build an explicit example, we recall that the minimal finite non-abelian simple groups are: $\text{PSL}_2(2^p)$ for any prime p , $\text{PSL}_2(3^p)$ for any odd prime p , $\text{PSL}_2(p)$ where $p > 3$ and 5 divides $p^2 + 1$, $\text{Sz}(2^p)$ for any odd prime p and $\text{PSL}_3(3)$. Let $(p_k)_{k \in \mathbb{N}}$ be any sequence of prime numbers without repeated terms. Then Corollary D yields that the infinitely iterated wreath product w.r.t. product actions of type $\mathcal{S} = (S_k)_{k \in \mathbb{N}}$, with $S_k = \text{PSL}_2(2^{p_k}) \leq \text{Sym}(2^{p_k} + 1)$ acting on the projective line over $\mathbb{F}_{2^{p_k}}$, is co-Hopfian.

2. PRELIMINARIES

In this section we collect some definitions that provide a more general context for our main theorems and serve as ingredients for the proofs.

2.1. Iterated wreath products. First we elaborate on the concept of an infinitely iterated wreath product. Let $\mathcal{S} = (S_k)_{k \in \mathbb{N}}$ be a sequence of finite groups. One can define, in many ways, a new sequence of permutation groups $\widehat{\mathcal{S}} = (\widehat{S}_k)_{k \in \mathbb{N}}$, with $\widehat{S}_k \leq \text{Sym}(\widehat{\Omega}_k)$, recursively as follows: (i) set $\widehat{S}_1 = S_1$ and choose a transitive faithful action of \widehat{S}_1 on a finite set $\widehat{\Omega}_1$; (ii) for $k \geq 2$, let \widehat{S}_k be the wreath product of S_k by \widehat{S}_{k-1} w.r.t. the given transitive faithful action of \widehat{S}_{k-1} and choose a transitive faithful action of \widehat{S}_k on a finite set $\widehat{\Omega}_k$. For each $k \in \mathbb{N}$, the group \widehat{S}_k is called a *k-fold iterated wreath product of type (S_1, \dots, S_k)* . The resulting sequence $\widehat{\mathcal{S}}$ constitutes in a natural way

an inverse system of finite groups; we call its inverse limit $\varprojlim \widehat{S}_k$ an *infinitely iterated wreath product of type \mathcal{S}* .

The infinitely iterated wreath products w.r.t. product actions, discussed in the introduction, fall into this scheme. We are, in fact, interested also in finitely iterated wreath products w.r.t. product actions and in iterated wreath products w.r.t. imprimitive actions. We employ the symbols $\textcircled{\cdot}$ and \wr to distinguish between wreath products w.r.t. product actions and imprimitive actions. Using notation that is chosen to fit our later applications (e.g., compare Proposition 3.1), we describe the two constructions as follows.

Definition 2.1. Let $\mathcal{S} = (S_k)_{k \in \mathbb{N} \cup \{0\}}$ be a sequence of finite permutation groups $S_k \leq \text{Sym}(\Omega_k)$, and set $\mathcal{S}' = (S_k)_{k \in \mathbb{N}}$.

(1) Define inductively $\widehat{\Omega}_1 = \Omega_1$ and $\widehat{\Omega}_n = \Omega_n \times \widehat{\Omega}_{n-1}$ for $n \geq 2$. The *n th iterated wreath product $W_n^{\text{ia}} \leq \text{Sym}(\widehat{\Omega}_n)$ of type $\mathcal{S}' = (S_1, \dots, S_n)$ w.r.t. imprimitive actions* is given by

$$\begin{aligned} W_1^{\text{ia}} &= W^{\text{ia}}(\mathcal{S}'_1) = S_1 \leq \text{Sym}(\widehat{\Omega}_1), \\ W_n^{\text{ia}} &= W^{\text{ia}}(\mathcal{S}'_n) = S_n \wr W_{n-1}^{\text{ia}} \leq \text{Sym}(\widehat{\Omega}_n) \quad \text{for } n \geq 2. \end{aligned}$$

The explicit realisation of the wreath product as a semidirect product is recalled in the proof of Proposition 3.3. The *infinitely iterated wreath product of type \mathcal{S}' w.r.t. imprimitive actions* is the inverse limit $W^{\text{ia}}(\mathcal{S}') = \varprojlim W_n^{\text{ia}}$ of the natural inverse system $W_1^{\text{ia}} \leftarrow W_2^{\text{ia}} \leftarrow \dots$.

(2) Define inductively $\widehat{\Omega}_0 = \Omega_0$ and $\widehat{\Omega}_n = \Omega_n^{\widehat{\Omega}_{n-1}}$ for $n \geq 1$. The *n th iterated wreath product $W_n^{\text{pa}} \leq \text{Sym}(\widehat{\Omega}_n)$ of type $\mathcal{S}_n = (S_0, \dots, S_{n-1})$ w.r.t. product actions* is given by

$$\begin{aligned} W_1^{\text{pa}} &= W^{\text{pa}}(\mathcal{S}_1) = S_0 \leq \text{Sym}(\widehat{\Omega}_0), \\ W_n^{\text{pa}} &= W^{\text{pa}}(\mathcal{S}_n) = S_{n-1} \textcircled{\cdot} W_{n-1}^{\text{pa}} \leq \text{Sym}(\widehat{\Omega}_{n-1}) \quad \text{for } n \geq 2. \end{aligned}$$

The explicit realisation of the wreath product as a semidirect product is recalled in the proof of Proposition 3.3. The *infinitely iterated wreath product of type \mathcal{S} w.r.t. product actions* is the inverse limit $W^{\text{pa}}(\mathcal{S}) = \varprojlim W_n^{\text{pa}}$ of the natural inverse system $W_1^{\text{pa}} \leftarrow W_2^{\text{pa}} \leftarrow \dots$.

2.2. Composition series. Let G be a countably based profinite group. Recall that every descending sequence $G = G_1 \supseteq G_2 \supseteq \dots$ of open subgroups with $\bigcap_n G_n = 1$ forms a neighbourhood basis of the identity element; see [15, Lemma 0.3.1(h)]. A *composition series* $(G_n)_{n \in \mathbb{N}}$ for G consists of open subnormal subgroups $G = G_1 \triangleright G_2 \triangleright \dots$ with $\bigcap_n G_n = 1$ and finite simple *composition factors* $S_n = G_n/G_{n+1}$ for $n \in \mathbb{N}$; we refer to $\mathcal{S} = (S_n)_{n \in \mathbb{N}}$ as a *sequence of composition factors* for G . The Jordan–Hölder Theorem for finite groups implies that any two composition series of G are equivalent in the sense that the composition factors (up to isomorphism) occur with the same multiplicities in both series.

3. EMBEDDING THEOREMS

In this section we establish the following basic fact which leads directly to a proof of Theorem A.

Proposition 3.1. *Let $\mathcal{S} = (S_k)_{k \in \mathbb{N} \cup \{0\}}$ be a sequence of finite simple groups, where each $S_k \leq \text{Sym}(S_k)$ forms a permutation group via the right regular action. Then the infinitely iterated wreath product $W^{\text{ia}}(\mathcal{S}')$ of type $\mathcal{S}' = (S_k)_{k \in \mathbb{N}}$ w.r.t. imprimitive actions embeds as a closed subgroup into the infinitely iterated wreath product $W^{\text{pa}}(\mathcal{S})$ of type $\mathcal{S} = (S_k)_{k \in \mathbb{N} \cup \{0\}}$ w.r.t. product actions.*

Proof of Theorem A. Set $\mathcal{S}' = (S_n)_{n \in \mathbb{N}}$, and let G be a countably based profinite group that admits \mathcal{S}' as a sequence of composition factors. Fix a composition series $G = G_1 \triangleright G_2 \triangleright \dots$ with $S_n \cong G_n/G_{n+1}$ for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, choose representatives $T_n = \{t_{s_n}^{(n)} \in G_n \mid s_n \in S_n\}$ for the cosets of G_{n+1} in G_n and, for $g \in G_n$, denote by $[g]_n \in T_n$ the representative of g modulo G_{n+1} . The set

$$\bigcup_{N \in \mathbb{N}_0} (T_N \times \dots \times T_2 \times T_1),$$

of finite words in the ‘alphabet’ $(T_n)_{n \in \mathbb{N}}$, forms a rooted spherically homogeneous tree \mathcal{T} with respect to the prefix partial order, whose layers are in natural correspondence with the finite coset spaces $G_n \backslash G$. As $\bigcap_n G_n = 1$, the group G acts faithfully on the boundary $\partial \mathcal{T}$, and hence on \mathcal{T} , via right multiplication: for $(t_{s_n}^{(n)})_{n \in \mathbb{N}} \in \partial \mathcal{T}$ and $g \in G$ the element $(u_n)_{n \in \mathbb{N}} = ((t_{s_n}^{(n)})_{n \in \mathbb{N}})^g \in \partial \mathcal{T}$ is given recursively by

$$g_1 = g, \quad u_n = [t_{s_n}^{(n)} g_n]_n \text{ for } n \geq 1, \quad g_n = g_{n-1} u_{n-1}^{-1} \text{ for } n \geq 2;$$

compare [4, proof of Theorem 2.6A].

This yields a continuous, hence closed embedding of the compact group G into the profinite group $\text{Aut}(\mathcal{T})$. Furthermore, by construction the image of G lies in a subgroup $W \leq \text{Aut}(\mathcal{T})$ that is naturally isomorphic to $W^{\text{ia}}(\mathcal{S}')$. Now, Proposition 3.1 shows that $W^{\text{ia}}(\mathcal{S}')$ and hence also G embed as closed subgroups into $W^{\text{pa}}(\mathcal{S})$. \square

The proof of Proposition 3.1 relies on a construction regarding finite wreath products. For any set X let $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$ denote the power set of X , and, for any given cardinal r , we write $\mathcal{P}_r(X) = \{Y \in \mathcal{P}(X) \mid |Y| = r\}$. A permutation group $G \leq \text{Sym}(\Sigma)$ has a natural induced permutation action on each $\mathcal{P}_r(\Sigma)$, via $\Gamma^g = \{\gamma^g \mid \gamma \in \Gamma\}$ for $\Gamma \subseteq \Sigma$ and $g \in G$.

Definition 3.2. Let $H \leq \text{Sym}(\Delta)$ and $G \leq \text{Sym}(\Sigma)$ be permutation groups. Consider the induced action of G on $\mathcal{P}_r(\Sigma)$ for some cardinal r . We say that H is *P-embedded of degree r* in G if there exist

- a collection $\tilde{\Delta} \subseteq \mathcal{P}_r(\Sigma)$ of pairwise disjoint sets and

- a subgroup $\tilde{H} \leq G$ such that $\tilde{\Delta}$ is \tilde{H} -invariant and the action of \tilde{H} on $\tilde{\Delta}$ is equivalent to the action of H on Δ .

In other words, $H \leq \text{Sym}(\Delta)$ is P-embedded of degree r in $G \leq \text{Sym}(\Sigma)$ if there exist an isomorphism $\iota: H \rightarrow \tilde{H} \leq G$ and a bijection $\Gamma: \Delta \rightarrow \tilde{\Delta} \subseteq \mathcal{P}_r(\Sigma)$ such that

$$\Gamma(\delta^h) = \Gamma(\delta)^{\iota(h)} \quad \text{for } \delta \in \Delta \text{ and } h \in H.$$

We remark that $H \leq \text{Sym}(\Delta)$ is P-embedded of degree 1 in $G \leq \text{Sym}(\Sigma)$ if and only if $H \leq \text{Sym}(\Delta)$ is *permutationally isomorphic to a subgroup* of $G \leq \text{Sym}(\Sigma)$ as described in the introduction.

Proposition 3.3. *Let $H \leq \text{Sym}(\Delta)$, $G \leq \text{Sym}(\Sigma)$ and $S \leq \text{Sym}(\Omega)$ be non-trivial finite permutation groups. Suppose that*

$$(3.1) \quad \iota: H \rightarrow \tilde{H} \leq G \quad \text{and} \quad \Gamma: \Delta \rightarrow \tilde{\Delta} \subseteq \mathcal{P}_r(\Sigma)$$

provide a P-embedding of degree $r \geq 2$.

Then the imprimitive wreath product $V = S \wr H \leq \text{Sym}(\Omega \times \Delta)$ can be P-embedded into the primitive wreath product $W = S \wr G \leq \text{Sym}(\Omega^\Sigma)$. More specifically, writing $\Phi = \Omega \times \Delta$, there are an integer $\hat{r} \geq 2$ and a P-embedding of degree \hat{r} via

$$\hat{\iota}: V \rightarrow \tilde{V} \leq W \quad \text{and} \quad \hat{\Gamma}: \Phi \rightarrow \tilde{\Phi} \subseteq \mathcal{P}_{\hat{r}}(\Omega^\Sigma),$$

such that $\hat{\iota}$ induces, upon factoring out the base groups on both sides, the original isomorphism ι .

Proof. We identify $W = S \wr G$ with $A \rtimes G$, where $A = S^\Sigma$ denotes the base group. Elements $(s_\sigma)_\sigma \in A$ and $g \in G$ operate on Ω^Σ by

$$f^{(s_\sigma)_\sigma}(\tau) = f(\tau)^{s_\tau} \quad \text{and} \quad f^g(\tau) = f(\tau^{g^{-1}}) \quad \text{for } f \in \Omega^\Sigma, \tau \in \Sigma.$$

Similarly we identify $V = S \wr H$ with $B \rtimes H$, where $B = S^\Delta$ denotes the base group. Elements $(s_\delta)_\delta \in B$ and $h \in H$ operate on $\Omega \times \Delta$ by

$$(\omega, \varepsilon)^{(s_\delta)_\delta} = (\omega^{s_\varepsilon}, \varepsilon) \quad \text{and} \quad (\omega, \varepsilon)^h = (\omega, \varepsilon^h) \quad \text{for } (\omega, \varepsilon) \in \Omega \times \Delta.$$

The P-embedding (3.1) yields a collection

$$\tilde{\Delta} = \{\Gamma(\delta) \mid \delta \in \Delta\} \subseteq \mathcal{P}_r(\Sigma)$$

of pairwise disjoint r -element subsets of Σ that are (i) in bijective correspondence with Δ and (ii) being permuted by $\tilde{H} \leq G$ in the same way as the elements of Δ are being permuted by H .

We define $\tilde{V} = \tilde{B} \rtimes \tilde{H} \leq W$, where $\tilde{B} \leq A$ denotes the image of B under the isomorphism

$$\iota': B \rightarrow \tilde{B}, \quad (s_\delta)_{\delta \in \Delta} \mapsto (t_\sigma)_{\sigma \in \Sigma}, \quad \text{where } t_\sigma = \begin{cases} s_\delta & \text{if } \sigma \in \Gamma(\delta), \\ 1 & \text{otherwise.} \end{cases}$$

A routine verification shows that ι and ι' induce together an isomorphism $\hat{\iota}: V \rightarrow \tilde{V}$ between groups.

Recall that $\min(|\Omega|, |\Delta|, r) \geq 2$ and that we write $\Phi = \Omega \times \Delta$. For

$$\hat{r} = |\Omega|^{|\Sigma| - r|\Delta|} \cdot (|\Omega|^r - |\Omega|)^{|\Delta| - 1} \geq 2$$

we obtain a bijection

$$\hat{\Gamma}: \Phi \rightarrow \tilde{\Phi} = \{\hat{\Gamma}(\varphi) \mid \varphi \in \Phi\} \subseteq \mathcal{P}_{\hat{r}}(\Omega^\Sigma)$$

by setting, for each $\varphi = (\omega, \varepsilon) \in \Phi$,

$$\begin{aligned} \hat{\Gamma}(\varphi) = \{f: \Sigma \rightarrow \Omega \mid f \text{ is constant and equal to } \omega \text{ on } \Gamma(\varepsilon), \text{ but} \\ f \text{ is not constant on any } \Gamma(\delta) \text{ for } \delta \in \Delta \text{ with } \delta \neq \varepsilon\}. \end{aligned}$$

Moreover, $\hat{\Gamma}(\varphi) \cap \hat{\Gamma}(\varphi') = \emptyset$ for all $\varphi, \varphi' \in \Phi$ with $\varphi \neq \varphi'$.

A routine calculation shows that, for $\varphi = (\omega, \varepsilon) \in \Phi$,

$$\hat{\Gamma}(\varphi^{(s_\delta)_\delta}) = \hat{\Gamma}(\omega^{s_\varepsilon}, \varepsilon) = \hat{\Gamma}(\omega, \varepsilon)^{\iota'((s_\delta)_\delta)} = \hat{\Gamma}(\varphi)^{\tilde{\iota}((s_\delta)_\delta)} \quad \text{for } (s_\delta)_\delta \in B$$

and

$$\hat{\Gamma}(\varphi^h) = \hat{\Gamma}(\omega, \varepsilon^h) = \hat{\Gamma}(\omega, \varepsilon)^{\iota(h)} = \hat{\Gamma}(\varphi)^{\hat{\iota}(h)} \quad \text{for } h \in H.$$

Thus $(\hat{\iota}, \hat{\Gamma})$ provides the required P-embedding. \square

We obtain the following corollary which in turn supplies a proof of Proposition 3.1.

Corollary 3.4. *Let $\mathcal{S} = (S_k)_{k \in \mathbb{N} \cup \{0\}}$, with $S_k \leq \text{Sym}(\Omega_k)$, be a sequence of non-trivial finite permutation groups, and set $\mathcal{S}' = (S_k)_{k \in \mathbb{N}}$.*

(1) *For every $n \in \mathbb{N}$, the n th iterated wreath product $W^{\text{ia}}(\mathcal{S}'_n)$ of type $\mathcal{S}'_n = (S_1, \dots, S_n)$ w.r.t. imprimitive actions is P-embedded in the $(n+1)$ th iterated wreath product $W^{\text{pa}}(\mathcal{S}_{n+1})$ of type $\mathcal{S}_{n+1} = (S_0, \dots, S_n)$ w.r.t. product actions.*

(2) *The P-embeddings can be chosen compatible with one another so that they induce an embedding of $W^{\text{ia}}(\mathcal{S}')$ into $W^{\text{pa}}(\mathcal{S})$ as a closed subgroup.*

4. NON-CO-HOPFIAN ITERATED WREATH PRODUCTS

In this section we prove Corollary B, Theorem C and Corollary D. Recall that being “permutationally isomorphic to a subgroup” of a permutation group is essentially the same as being P-embedded of degree 1.

Lemma 4.1. *For $i \in \{1, 2\}$ let $H_i \leq \text{Sym}(\Delta_i)$ and $G_i \leq \text{Sym}(\Omega_i)$ be non-trivial finite permutation groups, and suppose that*

$$\iota_i: H_i \rightarrow \tilde{H}_i \leq G_i \quad \text{and} \quad \gamma_i: \Delta_i \rightarrow \tilde{\Delta}_i \subseteq \Omega_i$$

provide permutation isomorphisms of H_i to subgroups of G_i .

Then the primitive wreath product $V = H_1 \wr H_2 \leq \text{Sym}(\Delta_1^{\Delta_2})$ is permutationally isomorphic to a subgroup of the primitive wreath product $W = G_1 \wr G_2 \leq \text{Sym}(\Omega_1^{\Omega_2})$. More specifically, writing $\Phi = \Delta_1^{\Delta_2}$, there is a permutation isomorphism via

$$\widehat{\iota}: V \rightarrow \widetilde{V} \leq W \quad \text{and} \quad \widehat{\gamma}: \Phi \rightarrow \widetilde{\Phi} \subseteq \Omega_1^{\Omega_2},$$

such that $\widehat{\iota}$ induces, upon factoring out the base groups on both sides, the original isomorphism $\iota_2: H_2 \rightarrow \widetilde{H}_2$.

Proof. Similar to the proof of Proposition 3.3 we identify $W = G_1 \wr G_2$ with $A \rtimes G_2$, where $A = G_1^{\Omega_2}$ denotes the base group, and $V = H_1 \wr H_2$ with $B \rtimes H_2$, where $B = H_1^{\Delta_2}$ denotes the base group. For notational simplicity we may assume, for $i \in \{1, 2\}$, that γ_i is just the identity map on $\Delta_i = \widetilde{\Delta}_i$.

We define $\widetilde{V} = \widetilde{B} \rtimes \widetilde{H}_2 \leq W$, where $\widetilde{B} \leq A$ denotes the image of B under the isomorphism $\iota': B \rightarrow \widetilde{B}$ given by

$$(h_\delta)_{\delta \in \Delta_2} \mapsto (g_\omega)_{\omega \in \Omega_2}, \quad \text{where} \quad g_\omega = \begin{cases} h_\omega & \text{if } \omega \in \Delta_2, \\ 1 & \text{otherwise.} \end{cases}$$

A routine verification shows that ι_2 and ι' induce an isomorphism $\widehat{\iota}: V \rightarrow \widetilde{V}$ of groups.

Recall that we write $\Phi = \Delta_1^{\Delta_2}$, and fix an arbitrary point $\alpha \in \Omega_1$. We obtain a bijection

$$\widehat{\gamma}: \Phi \rightarrow \widetilde{\Phi} = \{\widehat{\gamma}(f) \mid f \in \Phi\} \subseteq \Omega_1^{\Omega_2}$$

by setting, for each $f: \Delta_2 \rightarrow \Delta_1$ in Φ ,

$$\widehat{\gamma}(f) = \widetilde{f}: \Omega_2 \rightarrow \Omega_1, \quad \widetilde{f}(\omega) = \begin{cases} f(\omega) & \text{if } \omega \in \Delta_2, \\ \alpha & \text{otherwise.} \end{cases}$$

It is routine to verify that $(\widehat{\iota}, \widehat{\gamma})$ gives the required permutation isomorphism. \square

Lemma 4.2. *Let $H \leq \text{Sym}(\Delta)$, $K \leq \text{Sym}(\Psi)$ and $G \leq \text{Sym}(\Omega)$ be non-trivial finite permutation groups. Then the primitive wreath product $V = H \wr G \leq \text{Sym}(\Delta^\Omega)$ is permutationally isomorphic to a subgroup of the primitive wreath product $W = H \wr (K \wr G) \leq \text{Sym}(\Delta^{(\Psi^\Omega)})$. More specifically, writing $\Phi = \Delta^\Omega$ and identifying the top groups of V and W as usual with G , there is a permutation isomorphism via*

$$\iota: V \rightarrow \widetilde{V} \leq W \quad \text{and} \quad \gamma: \Phi \rightarrow \widetilde{\Phi} \subseteq \Delta^{(\Psi^\Omega)}$$

such that ι induces, upon factoring out the relevant base groups, the identity map between the top groups identified with G .

Proof. We identify $V = H \wr G$ with $B \rtimes G$, where $B = H^\Omega$ denotes the base group, and $U = K \wr G$ with $C \rtimes \tilde{G}$, where $C = K^\Omega$ denotes the base group and \tilde{G} is just a copy of G . We identify $W = H \wr U$ with $A \rtimes (C \rtimes \tilde{G})$, where $A = H^{\Psi^\Omega}$ denotes the base group.

Fix an element $\psi \in \Psi$. Setting, for $\omega \in \Omega$,

$$\Gamma(\omega) = \{f: \Omega \rightarrow \Psi \mid f(\omega) = \psi \text{ and } f \text{ is constant and} \\ \text{different from } \psi \text{ on } \Omega \setminus \{\omega\}\}$$

we obtain a P-embedding of degree $r = |\Psi| - 1$ of $G \leq \text{Sym}(\Omega)$ into $U \leq \text{Sym}(\Psi^\Omega)$ via

$$\iota' = \text{id}_G: G \rightarrow \tilde{G} \quad \text{and} \quad \Gamma: \Omega \rightarrow \tilde{\Omega} = \{\Gamma(\omega) \mid \omega \in \Omega\} \subseteq \mathcal{P}_r(\Psi^\Omega).$$

Next we define $\tilde{V} = \tilde{B} \rtimes \tilde{G} \leq W$, where $\tilde{B} \leq A$ denotes the image of B under the isomorphism

$$\iota'': B \rightarrow \tilde{B}, \quad (h_\omega)_{\omega \in \Omega} \mapsto (\tilde{h}_f)_{f \in \Psi^\Omega}, \quad \text{where } \tilde{h}_f = \begin{cases} h_\omega & \text{if } f \in \Gamma(\omega), \\ 1 & \text{otherwise.} \end{cases}$$

A routine verification shows that ι' and ι'' induce an isomorphism $\iota: V \rightarrow \tilde{V}$ of groups.

Recall that we write $\Phi = \Delta^\Omega$, and fix an arbitrary point $\alpha \in \Delta$. We obtain a bijection

$$\gamma: \Phi \rightarrow \tilde{\Phi} = \{\gamma(F) \mid F \in \Phi\} \subseteq \Delta^{(\Psi^\Omega)}$$

by setting

$$\gamma(F) = \tilde{F}: \Psi^\Omega \rightarrow \Delta, \quad \tilde{F}(f) = \begin{cases} F(\omega) & \text{for } f \in \Gamma(\omega), \\ \alpha & \text{otherwise.} \end{cases}$$

It is routine to verify that (ι, γ) gives the required permutation isomorphism. \square

Proposition 4.3. *Let $\mathcal{S} = (S_k)_{k \in \mathbb{N}}$ be a sequence of non-trivial finite permutation groups $S_k \leq \text{Sym}(\Omega_k)$ and let $\mathcal{S}^\circ = (S_{m(j)})_{j \in \mathbb{N}}$ for $m(1) < m(2) < \dots$ be a subsequence of \mathcal{S} .*

- (1) *For every $n \in \mathbb{N}$, the n th iterated wreath product $W^{\text{pa}}(\mathcal{S}_n^\circ)$ of type $\mathcal{S}_n^\circ = (S_{m(1)}, \dots, S_{m(n)})$ w.r.t. product actions is permutationally isomorphic to a subgroup of the $m(n)$ th iterated wreath product $W^{\text{pa}}(\mathcal{S}_{m(n)})$ of type $\mathcal{S}_{m(n)} = (S_1, \dots, S_{m(n)})$ w.r.t. product actions.*
- (2) *The permutation isomorphisms can be chosen compatible with one another so that they induce an embedding of $W^{\text{pa}}(\mathcal{S}^\circ)$ into $W^{\text{pa}}(\mathcal{S})$ as a closed subgroup.*

Proof. We prove (1) by induction on $n \in \mathbb{N}$.

For $n = 1$, it suffices to observe that $S_{m(1)}$ is permutationally isomorphic to a subgroup of the primitive wreath product $S_{m(1)} \wr W^{\text{pa}}(\mathcal{S}_{m(1)-1})$;

e.g., take the diagonal embedding of $S_{m(1)}$ into the base group acting on constant functions.

Now suppose that $n \geq 2$. By induction, $W^{\text{pa}}(\mathcal{S}_{n-1}^\circ)$ is permutationally isomorphic to a subgroup of $W^{\text{pa}}(\mathcal{S}_{m(n-1)})$. Repeated application of Lemma 4.2 shows that $W^{\text{pa}}(\mathcal{S}_n^\circ) = S_{m(n)} \wr W^{\text{pa}}(\mathcal{S}_{n-1}^\circ)$ is permutationally isomorphic to a subgroup of

$$S_{m(n)} \wr (S_{m(n-1)+l} \wr W^{\text{pa}}(\mathcal{S}_{m(n-1)+(l-1)})) = S_{m(n)} \wr W^{\text{pa}}(\mathcal{S}_{m(n-1)+l})$$

for $l \in \{1, \dots, m(n) - m(n-1) - 1\}$. The final value for l yields the requested permutation isomorphism to $W^{\text{pa}}(\mathcal{S}_{m(n)})$.

Claim (2) follows from the above construction and the compatibility assertion built into Lemma 4.2. \square

The proofs of Corollary B and Theorem C are now immediate.

Proof of Corollary B. Choose representatives X_1, X_2, \dots for the isomorphism types of finite simple groups in \mathcal{X} , and consider the sequence $\mathcal{S}_{\mathcal{X}}$ consisting of

$$X_1, X_1, X_2, X_1, X_2, X_3, \dots, X_1, X_2, \dots, X_n, \dots$$

Then every countably based profinite group with composition factors in \mathcal{X} has a composition series that forms, up to isomorphisms, a subsequence of $\mathcal{S}_{\mathcal{X}}$. Now apply Theorem A and Proposition 4.3. \square

Proof of Theorem C. Suppose that the terms of $\mathcal{S} = (S_k)_{k \in \mathbb{N}}$, with $S_k \leq \text{Sym}(\Omega_k)$, are eventually permutationally isomorphic to subgroups of later terms. Choose $n_0 \in \mathbb{N}$ and a strictly increasing, but non-identity function $m: \mathbb{N}_{>n_0} \rightarrow \mathbb{N}_{>n_0}$ such that: for each $j \in \mathbb{N}_{>n_0}$, the permutation group $S_j \leq \text{Sym}(\Omega_j)$ is permutationally isomorphic to a subgroup of $S_{m(j)} \leq \text{Sym}(\Omega_{m(j)})$.

Arguing similarly to the proof of Proposition 4.3 and applying, in addition, Lemma 4.1 in the induction step, we obtain an embedding of $W^{\text{pa}}(\mathcal{S})$ as a proper closed subgroup into itself. \square

Theorem C highlights two natural questions. Do there exist infinitely iterated wreath products of finite simple groups w.r.t. product actions that are co-Hopfian? To what extent are the hypotheses of Theorem C irredundant? Corollary D provides positive answers to both questions, when we restrict ourselves to minimal finite non-abelian simple groups.

Proof of Corollary D. In view of Theorem C only one implication remains to be shown. Suppose that \mathcal{S} consists of minimal finite non-abelian simple groups $S_k \leq \text{Sym}(\Omega_k)$, each equipped with a transitive permutation action and such that the terms of \mathcal{S} are not eventually permutationally isomorphic to a subgroup of a later terms. This means that, for every $n_0 \in \mathbb{N}$, there exists $k \geq n_0$ such that $S_k \leq \text{Sym}(\Omega_k)$ is permutationally isomorphic to a subgroup of, and hence equivalent to $S_j \leq \text{Sym}(\Omega_j)$ for only finitely many $j \in \mathbb{N}$. Denote by $G = W^{\text{pa}}(\mathcal{S})$

the infinitely iterated wreath product of type \mathcal{S} w.r.t. product actions. Suppose further that $H \leq_c G$ with $H \cong G$. We need to show that $H = G$.

Since the action of each S_j on Ω_j is transitive, it is easy to see that the open normal subgroups of G form a descending chain $G = N_0 \supseteq N_1 \supseteq \dots$, where

$$N_l = \ker(G \rightarrow W^{\text{pa}}(\mathcal{S}_l)) \quad \text{for } l \in \mathbb{N}.$$

The group $H \cong G$ has a corresponding chain of open normal subgroups $H = M_0 \supseteq M_1 \supseteq \dots$

For every $n \in \mathbb{N}$ we choose $m(n) \in \mathbb{N}$ such that $H \cap N_n = M_{m(n)}$ and observe that $G/N_{m(n)} \cong H/M_{m(n)} \cong HN_n/N_n \leq G/N_n$ implies $m(n) \leq n$. Moreover, it is enough to show that $m(n) \geq n$, hence $n = m(n)$, for infinitely many $n \in \mathbb{N}$; for this implies $HN_n = G$ for infinitely many n , and thus $H = G$.

Now, start with any large number $n_0 \in \mathbb{N}$. By our hypotheses, there is a $k \geq n_0$ such that

$$n = \min\{j \in \mathbb{N} \mid S_k \leq \text{Sym}(\Omega_k) \text{ is not equivalent to } S_l \leq \text{Sym}(\Omega_l) \text{ for any } l > j\} \in \mathbb{N}_{\geq n_0}$$

is finite. Clearly, the set of composition factors of N_n is $\{S_j \mid j > n\}$. In particular, the group N_n does not have any composition factors isomorphic to S_k .

Set $m = m(n)$ and assume, for a contradiction, that $m < n$. Then $N_n \geq H \cap N_n = M_m \geq M_{n-1}$. Observe that there exist $K \trianglelefteq_o M_{n-1}$ such that $X = M_{n-1}/K$ is isomorphic to S_k . We claim that X is also a composition factor of N_n . Indeed, by intersecting a composition series for N_n (possessing exclusively minimal non-abelian simple factors) with M_{n-1} we obtain a subnormal series of M_{n-1} with factors that are either soluble or isomorphic to a composition factor of N_n . This implies that each composition factor of M_{n-1} is either soluble or isomorphic to a composition factor of N_n . Consequently, $X \cong S_k$ is isomorphic to a composition factor of N_n , a contradiction. \square

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